Chapter 8: Modeling chains of ODEs

$$\frac{\mathrm{d}R}{\mathrm{d}t} = [r(1-R/K) - bN]R \ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = [bR - d - cM]N \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}t} = [cN - e]M$$

1 2 1

n=1 $\bar{R} = K$

n=2 $\bar{R} = \dfrac{d}{b}$ and $\bar{N} = \dfrac{r}{b}\left(1 - \dfrac{d}{bK}\right) = \dfrac{r}{b}\left(1 - \dfrac{1}{R_0}\right)$

Modeling chains of ODEs

$$\frac{\mathrm{d}R}{\mathrm{d}t} = [r(1 - R/K) - bN]R\ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = [bR - d - cM]N \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}t} = [cN - e]M$$

3

2

1

n=1 $\bar{R} = K$

n=2 $\bar{R} = \frac{d}{b} \quad \text{and} \quad \bar{N} = \frac{r}{b}\left(1 - \frac{d}{bK}\right) = \frac{r}{b}\left(1 - \frac{1}{R_0}\right)$

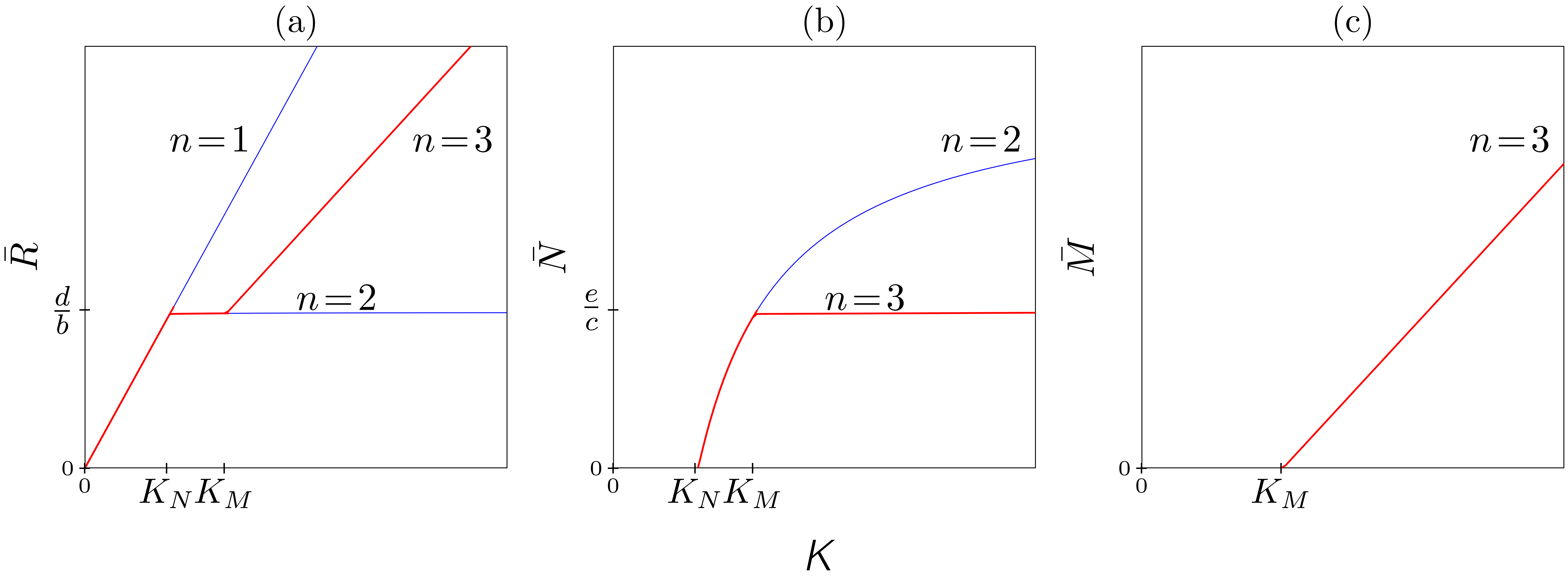
$$R_0' = \frac{cr}{be}$$

n=3 $\bar{N} = \frac{e}{c}\ , \bar{R} = K\left(1 - \frac{be}{cr}\right) \quad \text{and} \quad \bar{M} = \frac{b\bar{R} - d}{c}$

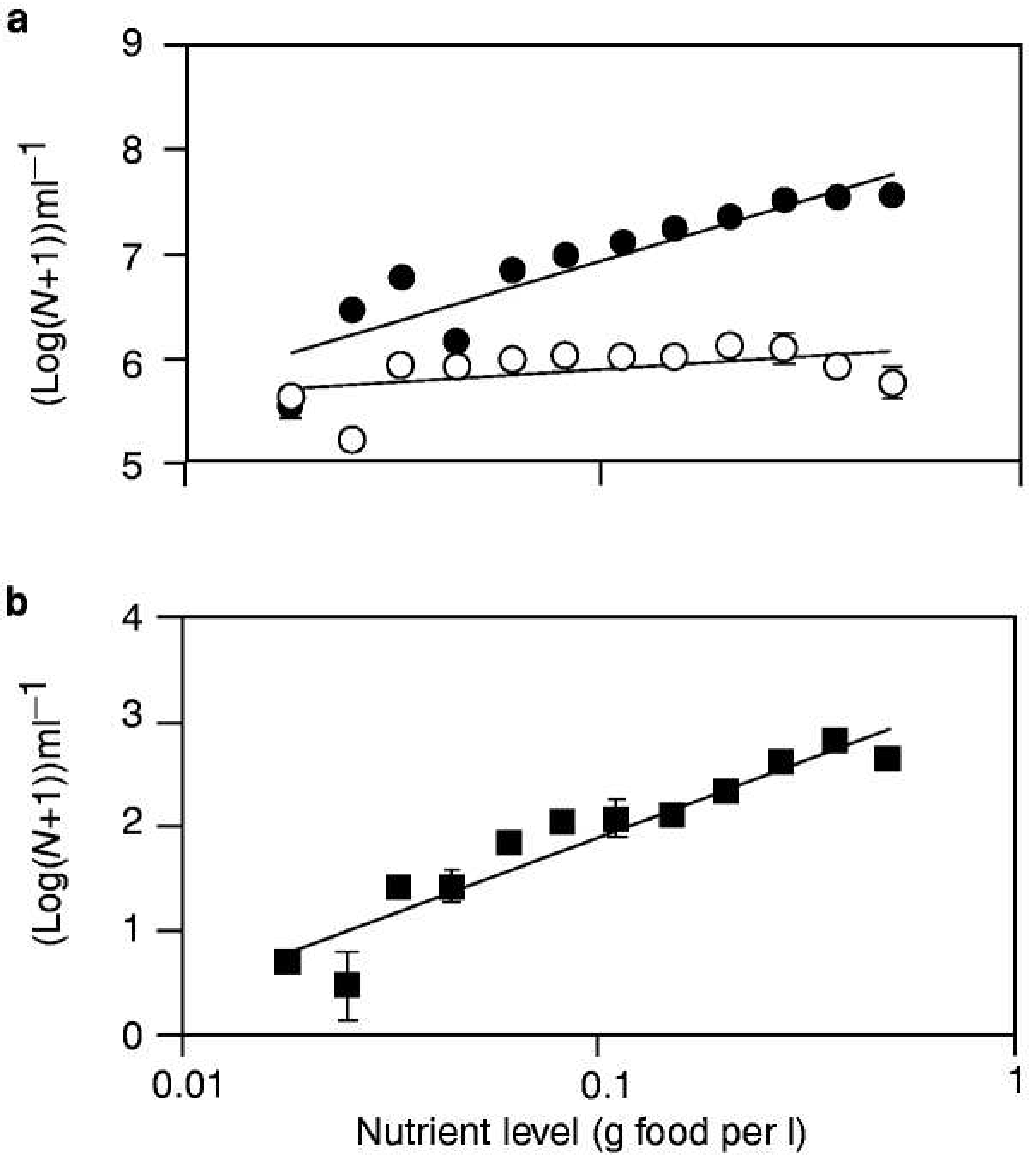
For odd chain lengths *R* depends on *K*

Modeling chains of ODEs

$$\frac{\mathrm{d}R}{\mathrm{d}t} = [r(1 - R/K) - bN]R\ , \quad \frac{\mathrm{d}N}{\mathrm{d}t} = [bR - d - cM]N \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}t} = [cN - e]M$$



Kaunzinger & Morin, Nature, 1998



Predator

Prey with predator

Prey alone

Bacterial food chain

Modeling chains with saturated interaction terms

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left[r\left(1 - \frac{R}{K}\right) - \frac{bN}{h_R + R}\right] R\,, \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \left[\frac{bR}{h_R + R} - d - \frac{cM}{h_N + N}\right] N\,, \text{ and } \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \left[\frac{cN}{h_N + N} - e\right] M$$

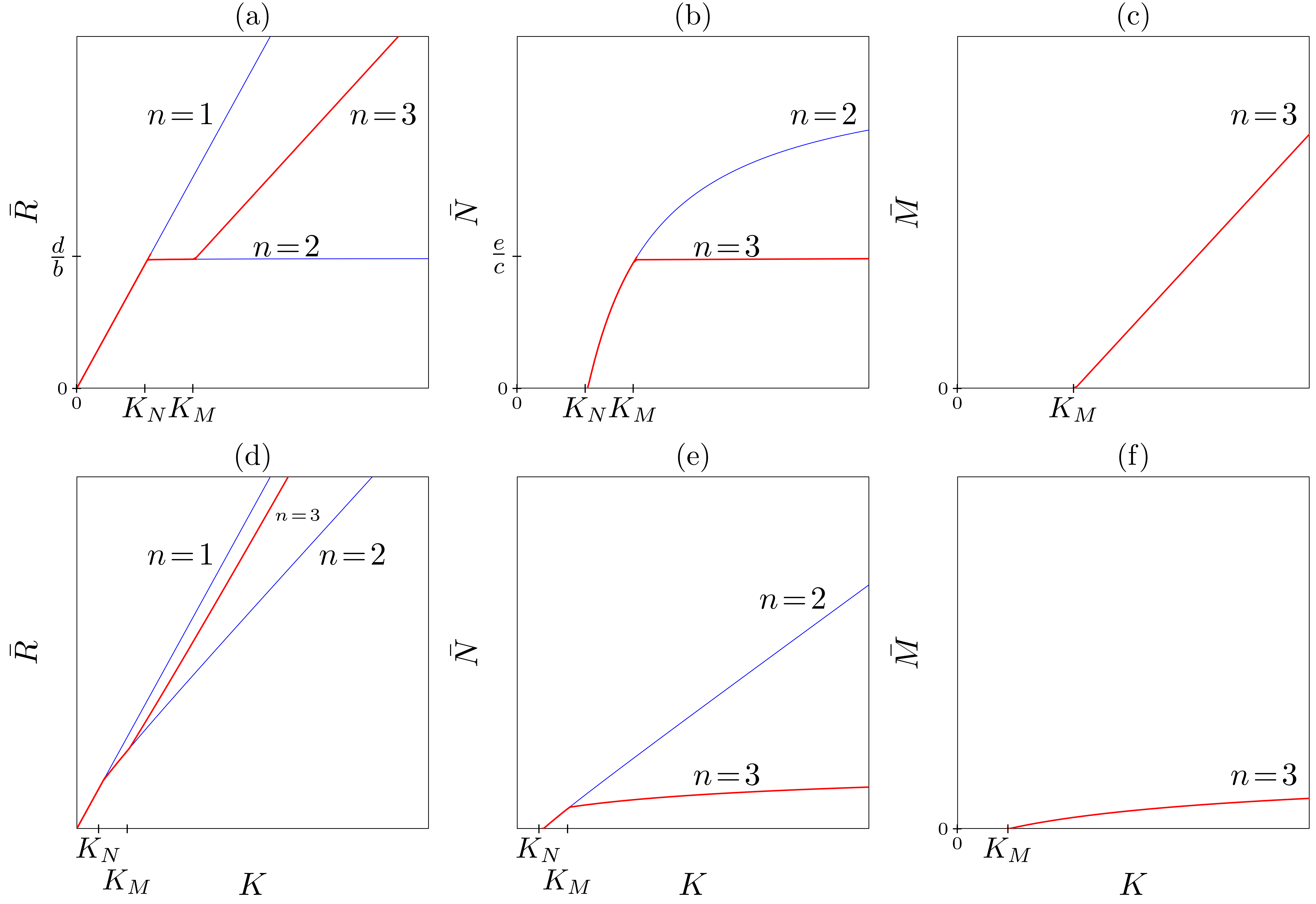
f_R: R and N *f_N: in absence of M no N* *f_M: no M*

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left[r\left(1 - \frac{R}{K}\right) - \frac{bN}{h_R + R + N}\right] R\,, \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \left[\frac{bR}{h_R + R + N} - d - \frac{cM}{h_N + N + M}\right] N \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}t} = \left[\frac{cN}{h_N + N + M} - e\right] M$$

Per capita function always depends on variable itself.

$$aXY \simeq \frac{aXY}{1 + X/k + Y/k} \quad \text{when } k \text{ is large}$$

Modeling chains with Beddington interaction terms



Other famous chains don't suffer from this problem

SEIR model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta SI \ , \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \beta SI - (d+\gamma)E \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \gamma E - (\delta + r)I \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR$$

$$\bar{R} = \frac{r}{d}\ \bar{I} \ , \quad \bar{I} = \frac{\gamma}{\delta + r}\ \bar{E} \ , \quad \bar{S} = \frac{(d+\gamma)(\delta + r)}{\gamma\beta} \ , \quad \bar{E} = \frac{s}{d+\gamma} - \frac{d(\delta + r)}{\gamma\beta}$$

$\bar{R}$ and $\bar{I}$ are proportional to previous level

Cascade of cell divisions

$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = s - (p+d)N_0 \ , \quad \frac{\mathrm{d}N_i}{\mathrm{d}t} = 2pN_{i-1} - (p+d)N_i \quad \text{and} \quad \frac{\mathrm{d}N_n}{\mathrm{d}t} = 2pN_{n-1} - dN_n \ ,$$

$$\bar{N}_0 = \frac{s}{p+d} \ , \quad \bar{N}_i = \frac{2p}{p+d}\ \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d}\ \bar{N}_{n-1}$$

$$J = \begin{pmatrix} -(p+d) & 0 & 0 & 0 & \dots & \dots & 0 \\ 2p & -(p+d) & 0 & 0 & \dots & \dots & 0 \\ 0 & 2p & -(p+d) & 0 & \dots & \dots & 0 \\ & & \vdots & & & & \\ 0 & \dots & 0 & \dots & 0 & 2p & -d \end{pmatrix}$$

$$(J_{00} - \lambda)(J_{11} - \lambda)(J_{22} - \lambda)\dots(J_{nn} - \lambda) = 0$$

Solve eigenvalues from determinant (product diagonal elements)

Cascade of cell divisions

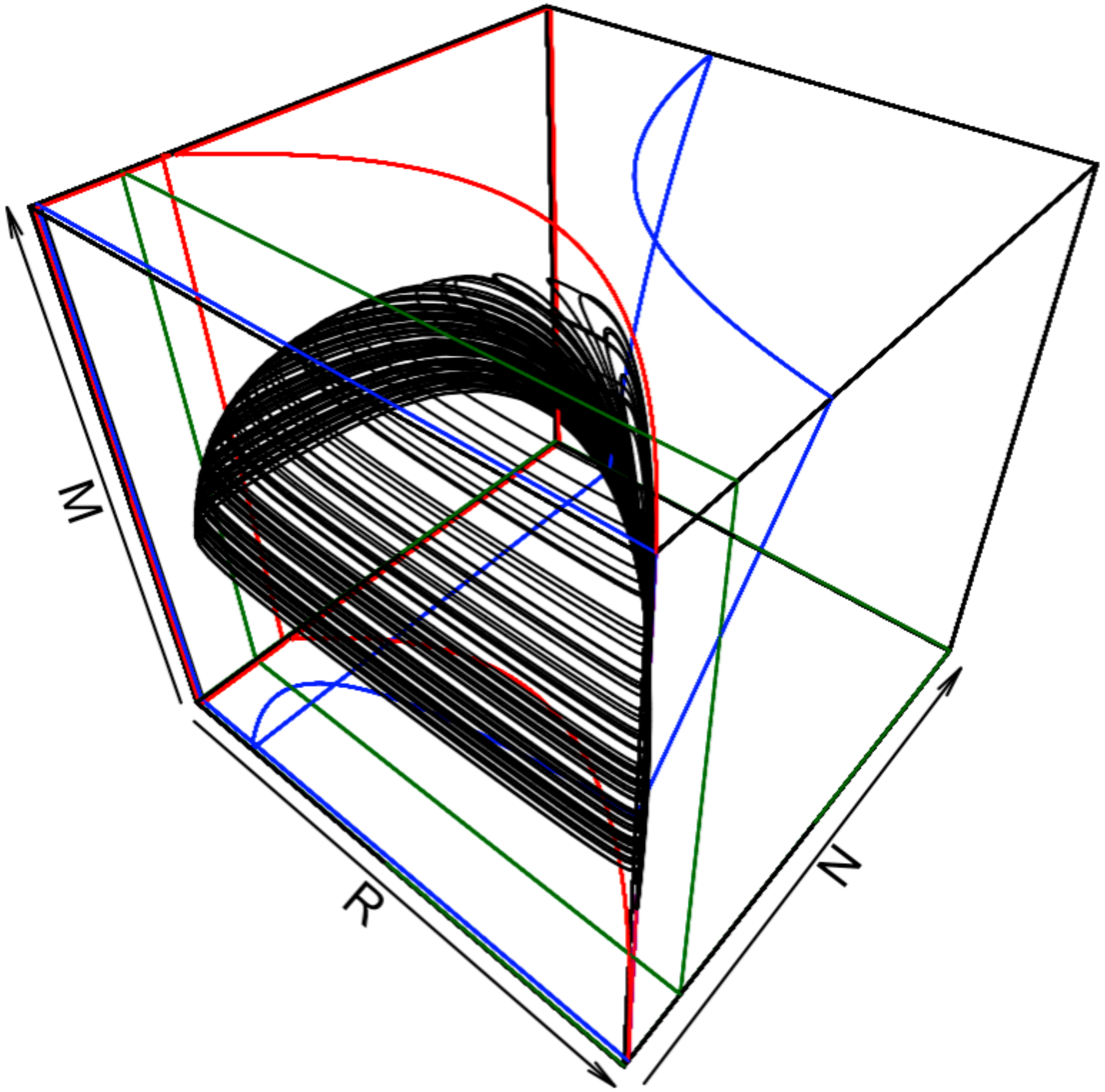
$$\frac{\mathrm{d}N_0}{\mathrm{d}t} = s - (p+d)N_0 \ , \quad \frac{\mathrm{d}N_i}{\mathrm{d}t} = 2pN_{i-1} - (p+d)N_i \quad \text{and} \quad \frac{\mathrm{d}N_n}{\mathrm{d}t} = 2pN_{n-1} - dN_n \ ,$$

$$\bar{N}_0 = \frac{s}{p+d} \ , \quad \bar{N}_i = \frac{2p}{p+d} \ \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d} \ \bar{N}_{n-1}$$

$$\bar{N}_0 = \frac{s}{p+d} \ , \quad \bar{N}_i = \frac{2^i p^i s}{(p+d)^{i+1}} \quad \text{and} \quad \bar{N}_n = \frac{s}{d} \ \left(\frac{2p}{p+d}\right)^n$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -aQ - d_Q Q + d \sum f_i N_i \quad \text{and} \quad s = aQ$$

Chaos in a 3D food chain



$$\frac{dR}{dt} = R(1-R) - c_1 N f(R) \ ,$$

$$\frac{dN}{dt} = -a_N N + c_1 N f(R) - c_2 M g(N) \ ,$$

$$\frac{dM}{dt} = -a_M M + c_2 M g(N) \ ,$$

Exercise 8.4

Kinetic proofreading: last exercise

Michaelis Menten:

$$F + L \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \quad \text{or} \quad \frac{\mathrm{d}C}{\mathrm{d}t} = k_1 F L - k_{-1} C \quad \text{with} \quad F = R - C \quad \text{gives} \quad C = \frac{RL}{K_m + L}$$

Kinetic proofreading:

$$F + L \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_0 \;, \qquad C_{i-1} \overset{k_2}{\rightarrow} C_i \quad \text{and} \quad C_i \overset{k_{-1}}{\rightarrow} F$$

$$\frac{\mathrm{d}C_0}{\mathrm{d}t} = k_1 F L - (k_{-1} + k_2) C_0 \;, \quad \frac{\mathrm{d}C_i}{\mathrm{d}t} = k_2 C_{i-1} - (k_{-1} + k_2) C_i \quad \text{and} \quad \frac{\mathrm{d}C_n}{\mathrm{d}t} = k_2 C_{n-1} - k_{-1} C_n$$

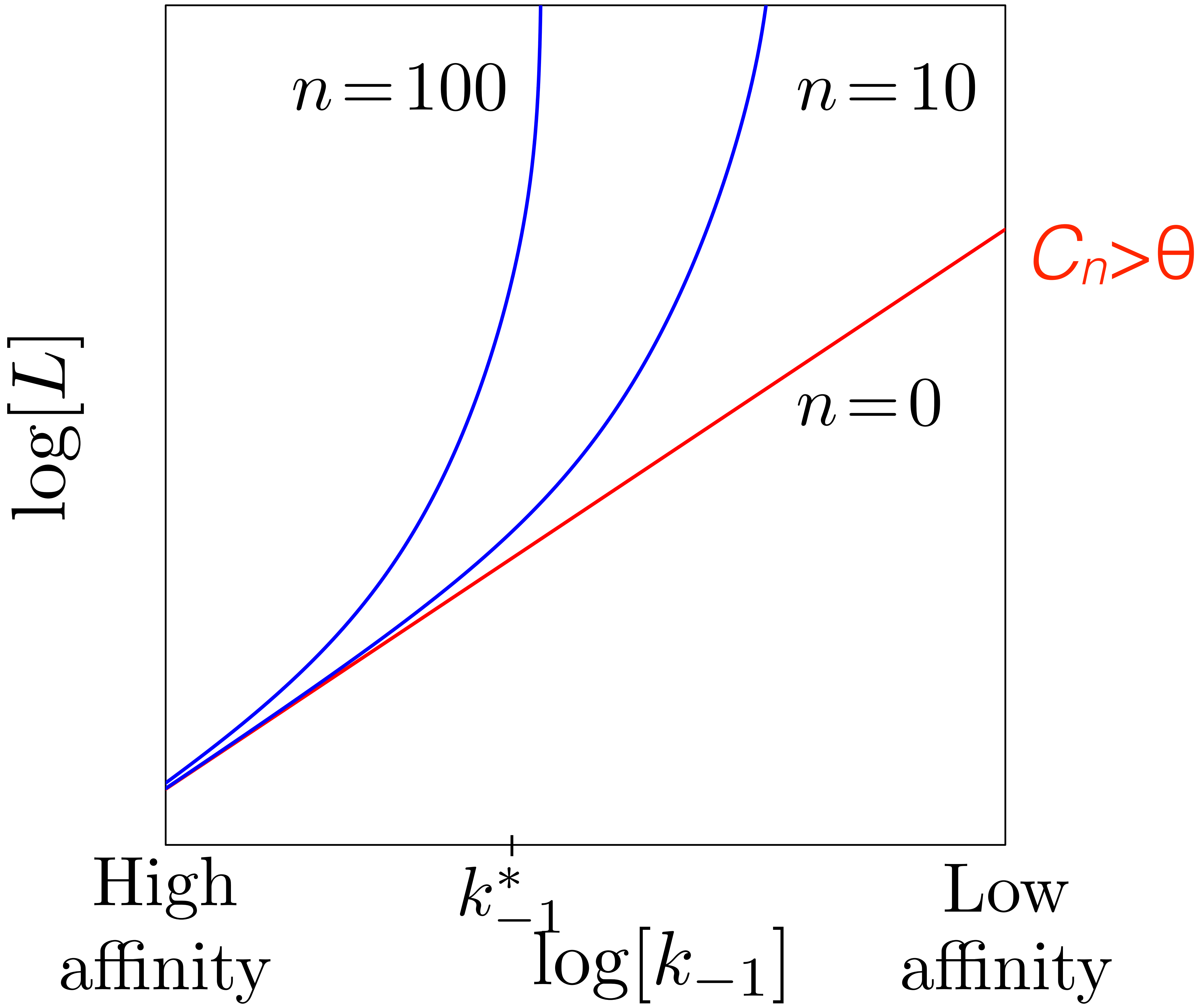
with $F = R - \sum_i^n C_i$ gives $\bar{C}_n = \dfrac{RL}{K_m + L} \left(\dfrac{k_2}{k_{-1} + k_2} \right)^n$

When $k_{-1} \ll k_2$ similar to Michaelis Menten, otherwise discrimination

Kinetic proofreading: last exercise

$$F + L \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_0 \ , \quad C_{i-1} \xrightarrow{k_2} C_i \quad \text{and} \quad C_i \xrightarrow{k_{-1}} F \quad \text{where} \quad F = R - C$$

$$\frac{\mathrm{d}C_0}{\mathrm{d}t} = k_1 F L - (k_{-1} + k_2) C_0 \ , \quad \frac{\mathrm{d}C_i}{\mathrm{d}t} = k_2 C_{i-1} - (k_{-1} + k_2) C_i \quad \text{and} \quad \frac{\mathrm{d}C_n}{\mathrm{d}t} = k_2 C_{n-1} - k_{-1} C_n$$



$$\bar{C}_n = \frac{RL}{K_m + L} \left(\frac{k_2}{k_{-1} + k_2} \right)^n$$