Chapter 6: R_0

SIR model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta S I \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - (\delta + r) I \ , \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ ,$$

Virulence: $v = \delta - d$

$$\bar{S} = \frac{\delta + r}{\beta} \ , \quad \bar{I} = \frac{s}{\delta + r} - \frac{d}{\beta} \ , \quad \text{and} \quad \bar{R} = \frac{r}{d} \ \bar{I} = \frac{rs}{d(\delta + r)} - \frac{r}{\beta} \ ,$$

$$R_0 = \beta \bar{S} \ \frac{1}{\delta + r} = \frac{s}{d} \ \frac{\beta}{\delta + r}$$

R_0 can also be computed from Jacobian

SIR model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta SI \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - (\delta + r)I \ , \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ ,$$

$$R_0 = \beta \bar{S} \, \frac{1}{\delta + r} = \frac{s}{d} \, \frac{\beta}{\delta + r}$$

SI model (I=0): $J = \begin{pmatrix} -d & -\beta \bar{S} \\ 0 & \beta \bar{S} - \delta - r \end{pmatrix}$

$\lambda_1 < 0$ same as $R_0 > 1$

$$\lambda_1 = \beta \bar{S} - \delta - r \ \text{and} \ \lambda_2 = -d.$$

R_0 is not equal to the growth/expansion rate of the epidemic

SIR model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta SI \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - (\delta + r)I \ , \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ , \quad R_0 = \beta \bar{S} \ \frac{1}{\delta + r} = \frac{s}{d} \ \frac{\beta}{\delta + r}$$

Initial expansion rate ($r_0 > 0$):

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta \bar{S} I - (\delta + r)I = \left(\frac{\beta s}{d} - \delta - r \right) I = r_0 I \ ,$$

$$\lambda_1 \ = \ \beta \bar{S} - \delta - r$$

R_0 is not equal to the growth/expansion rate of the epidemic

Initial expansion rate ($\rho_0 > 0$):

$$R_0 = \beta\bar{S}\,\frac{1}{\delta + r} = \frac{s}{d}\,\frac{\beta}{\delta + r}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta\bar{S}I - (\delta + r)I = \left(\frac{\beta s}{d} - \delta - r\right)I = \rho_0 I$$

Define length of the infectious period, $L = \dfrac{1}{\delta + r}$ to see that

$$R_0 = \beta\bar{S}L \quad \text{and} \quad \rho_0 = \beta\bar{S} - \frac{1}{L} \quad \text{such that} \quad \beta\bar{S} = \rho_0 + \frac{1}{L} \quad \text{and} \quad R_0 = \rho_0 L + 1$$

From the latter:

$$\rho_0 = \frac{R_0 - 1}{L} = (R_0 - 1)(\delta + r)$$

Frequency dependent infections

SIR model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta SI \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta SI - (\delta + r)I \ , \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ ,$$

$$R_0 = \frac{s}{d}\frac{\beta}{\delta + r}$$

Frequency dependent infection in SIR model: $N = S + I + R$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \frac{\beta SI}{N} \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta SI}{N} - (\delta + r)I \ , \quad \text{and} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ ,$$

$$R_0 = \frac{\beta}{\delta + r}$$

SEIR model

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta SI \ , \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \beta SI - (\gamma + d)E \ , \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \gamma E - (\delta + r)I \ , \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR \ ,$$

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$$\bar{I} = \frac{s}{\beta \bar{S}} - \frac{d}{\beta} = \frac{\gamma}{\gamma + d} \frac{s}{\delta + r} - \frac{d}{\beta} \leftarrow \bar{S} = \frac{\gamma + d}{\gamma} \frac{\delta + r}{\beta} \leftarrow \bar{E} = \frac{\delta + r}{\gamma} I$$

$$R_0 = \frac{s}{d} \frac{\beta}{\delta + r} \frac{\gamma}{\gamma + d} \ .$$

$R_0 > 1$ is the same as $\bar{I} > 0$

SEIR model

$$R_0 = \frac{s}{d}\,\frac{\beta}{\delta + r}\,\frac{\gamma}{\gamma + d}\,.$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = s - dS - \beta S I\,, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \beta S I - (\gamma + d)E\,, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \gamma E - (\delta + r)I\,, \quad \frac{\mathrm{d}R}{\mathrm{d}t} = rI - dR\,,$$

$$J = \begin{pmatrix} \partial_E E' & \partial_I E' \\ \partial_E I' & \partial_I I' \end{pmatrix} = \begin{pmatrix} -(\gamma + d) & \beta \bar{S} \\ \gamma & -(\delta + r) \end{pmatrix}$$

tr < 0 and det = $(\gamma + d)(\delta + r) - \gamma\beta\bar{S}$

$$\det J < 0 \quad \text{or} \quad \frac{d}{s}\,\frac{\gamma + d}{\gamma}\,\frac{\delta + r}{\beta} - 1 < 0 \quad \leftrightarrow \quad \frac{1}{R_0} < 1 \quad \leftrightarrow \quad R_0 > 1$$

Fitness in consumer-resource models

$$\frac{\mathrm{d}R}{\mathrm{d}t} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = caRN - \delta N$$

$$K = k(1-d/b) \qquad R_{0_R} = b/d. \qquad R_{0_N} = caK/\delta$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\beta RN}{h+R} - \delta N$$

$$R_0 = \frac{\beta K}{\delta(h+K)} \quad \text{or} \quad R_0 = \beta/\delta$$

$$K = k(1-d/b)$$